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INTEGRAL EXTENSIONS OF NONCOMMUTATIVE RINGS

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ABSTRACT
In this paper we study integral extensions of noncommutative rings. To begin, we prove that finite subnormalizing extensions are integral. This is done by proving a generalization of the Paré–Schelter result that a matrix ring is integral over the coefficient ring. Our methods are similar to those of Lorenz and Passman, who showed that finite normalizing extensions are integral. As corollaries we note that the (twisted) smash product over the restricted enveloping algebra of a finite dimensional restricted Lie algebra is integral over the coefficient ring and then prove a Going Up theorem for prime ideals in these ring extensions.

Next we study automorphisms of rings. In particular, we prove an integrality theorem for algebraic automorphisms. Combining group gradings and actions, we show that if a ring R is graded by a finite group G, and H is a finite group of automorphisms of R that permute the homogeneous components, with the order of H invertible in R, then R is integral over R_1^H, the fixed ring of the identity component. This, in turn, is used to prove our final result: Suppose that if H is a finite dimensional semisimple cocommutative Hopf algebra over an algebraically closed field of positive characteristic. If R is an H-module algebra, then R is integral over R^H, its subring of invariants.

Introduction

For noncommutative rings there are two related definitions of an integral extension, as follows: Let R be a ring containing a subset S. If $r \in R$, an *S-monomial in r* is a product each of whose factors is either r or an element of S, with at least one factor from S. The *degree* of this monomial is the number of factors "r" occurring in it. If $r_1, r_2, \ldots, r_m \in R$, an *S-monomial in the r_i* is a product each of whose factors is either one of the r_i or an element from S, with again at least one factor from S appearing. Here the *degree* is the total number of factors from the r_i occurring. We say that R is *Schelter integral over S* if, given $r \in R$, there exists

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an integer n such that $r^n = \phi$, where ϕ is a sum of S-monomials in r of degree less than n. R is said to be *fully integral of degree m over S* if, given $r_1, r_2, \ldots, r_m \in R$, we get that $r_1 r_2 \cdots r_m = \psi$, where ψ is a sum of S-monomials in the r_i of degree less than m. Notice that by setting $r_1 = r_2 = \cdots = r_m = r \in R$, it follows that full integrality implies Schelter integrality of bounded degree. It is usual to only define integrality for S a subring of R, but in the course of proofs it is sometimes convenient to allow more general subsets.

First, we consider finite subnormalizing ring extensions and prove that these are fully integral. Lorenz and Passman [LP] have proved this result for finite normalizing extensions. Our approach is based on theirs, which used a variation on the Paré–Schelter result [PS] that a matrix ring is Schelter integral over the coefficient ring. We prove here that it is sufficient to consider a suitable subring of lower triangular matrices. As a corollary we note that a twisted smash product of a restricted enveloping algebra of a finite dimensional restricted Lie algebra is fully integral over the coefficient ring. A Going Up theorem is then proved for prime ideals in these smash products.

We continue by considering automorphisms of rings. In [Q2] we proved that if G is a finite group of automorphisms of a ring R and the order of G is a unit in R, then R is fully integral over the fixed ring R^G. In Theorem 7 we extend this to an algebraic automorphism by first considering linear combinations of automorphisms.

Next we consider group gradings and actions together. Suppose that R is a ring graded by a finite group G, and H is a finite group of automorphisms of R which permute the homogeneous components. It follows that the elements of H act as automorphisms of R_1, the identity component of R. Theorem 8 shows that under these circumstances, if the order of H is invertible in R, then R is fully integral over R_1^H. If H is the trivial group then this is a result of G. Bergman (see [P1]). When G is trivial, this is [Q2, Theorem 1.3].

Finally, we use Theorem 8 to study certain Hopf algebra actions. Let H be a finite dimensional cocommutative semisimple Hopf algebra over an algebraically closed field of positive characteristic. If R is an H-module algebra we prove that R is fully integral over the invariant subring R^H. This is achieved by using structure theorems of Kostant and Sweedler to see that the problem reduces to a case of Theorem 8.

Subnormalizing extensions and integrality

Let $R \subseteq S$ be rings and suppose that there are finitely many elements $x_1, x_2, \ldots, x_n \in S$ such that $S = Rx_1 + Rx_2 + \cdots + Rx_n$ and $Rx_1 + Rx_2 + \cdots + Rx_i =$

$x_1R + x_2R + \cdots + x_iR$, for $i = 1,2,\ldots,n$. Then we say that S is a *finite subnormalizing extension* of R. These extensions were studied in [W]. If $Rx_i = x_iR$ for each i, then S is called a *finite normalizing extension.* In [LP], Lorenz and Passman showed that if S is a finite normalizing extension of R then S is integral over R. We extend this in Theorem 3 to finite subnormalizing extensions. First we need a technical result. This is a generalization, in the same spirit as [LP, Theorem 1] and [P1, Theorem 1], of the Paré–Schelter integrality theorem [PS, Theorem 1]. These generalizations use variations on the original proof. Here we follow [P2, Theorem 25.1] and therefore omit some of the details.

THEOREM 1. *Let R be a ring, possibly without identity. Suppose that $D \subseteq A \subseteq M_n(R)$ are rings, with D consisting of lower triangular matrices, and assume that for $i = 1,2,\ldots,n$,*

(i) $Ae_{i,i} \subseteq A$,

(ii) $e_{i,i}De_{i,i} = e_{i,i}Ae_{i,i}$.

Then A is fully integral over D of degree $m(n)$. Here m is a function of n.

PROOF. We need only show that A is Schelter integral over D of bounded degree $m(n)$ and then full integrality will follow as in [P2, Theorem 25.1]. Let A_k denote the set Af_k, where $f_k = e_{1,1} + e_{2,2} + \cdots + e_{k,k}$. From (i) we know that $Af_k \subseteq A$. The proof proceeds by induction on k. If $k = 1$ and $a \in A_1$, we can choose $t \in D$ such that $a^2 = at$. Therefore, a is Schelter integral over D of degree 2. Now suppose that we know that every element of A_{k-1} is Schelter integral over D of degree m and let $a \in A_k$. We partition a as

$$a = \begin{bmatrix} a' & a'' & 0 \\ A & b & 0 \\ C & D & 0 \end{bmatrix},$$

where a' is a $(k-1) \times (k-1)$ block and $b \in R$. Let

$$\tilde{a} = \begin{bmatrix} a' & 0 & 0 \\ A & 0 & 0 \\ C & 0 & 0 \end{bmatrix} \in A_{k-1}.$$

If $a_1, a_2 \in A_k$, we say $a_1 \equiv a_2$, if $a_1'' = a_2''$.

By analogy with [P2, Theorem 25.1] we have three facts:

(1) Suppose $a, a_1, a_2 \in A_k$ with $a_1 \equiv a_2$, and $t \in D$, then $ta_1 \equiv ta_2$ and $\tilde{a}a_1 \equiv \tilde{a}a_2$.

(2) If $a_1, a \in A_k$, then there exists $t \in D$ such that $\tilde{a}_1 a \equiv a_1 a - a_1 t$.

(3) If $a_1, a_2, \ldots, a_j$, $a \in A_k$, then $\tilde{a}_j\tilde{a}_{j-1}\cdots\tilde{a}_1 a \equiv a_j a_{j-1}\cdots a_1 a + \tau$, where τ is

a sum of nonconstant D-monomials in $a_1, a_2, \ldots, a_j$ of degree at most j. (We remark that in adapting the arguments given in [P2, Theorem 25.1] one needs the inclusions $TA_k \subseteq A_k$ and $A_kT \subseteq A_k$.)

Now we can finish the proof by induction. Let $a \in A_k$. Then $\tilde{a} \in A_{k-1}$ so that by our assumption $\tilde{a}$ satisfies an equation $\tilde{a}^m = \phi(\tilde{a})$, where ϕ is a sum of T-monomials in $\tilde{a}$ of degree less than m. Multiply on the right by $\tilde{a}$, to get an equation $\tilde{a}^{m+1} = \psi(\tilde{a})$, where each T-monomial in ψ ends in $\tilde{a}$. Consider $(\tilde{a}^{m+1} - \psi(\tilde{a}))a$. Each monomial in this expression is a product of at most $m + 2$ factors, where all but the last is either $\tilde{a}$ or of the form $t\tilde{a} = (ta)\tilde{}$, with $t \in D$. The last factor in each is a, of course.

By applying (3) above to each monomial in $(\tilde{a}^{m+1} - \psi(\tilde{a}))a$ we conclude that $0 \equiv (\tilde{a}^{m+1} - \psi(\tilde{a}))a \equiv (a^{m+1} - \psi(a))a + \tau$, where τ is a sum of D monomials in a of degree at most $m + 1$. Let $s = a^{m+2} - \psi(a)a + \tau$. Then

$$s = \begin{pmatrix} s' & 0 & 0 \\ A & b & 0 \\ C & D & 0 \end{pmatrix}.$$

Since

$$w = \begin{pmatrix} s' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in A_k,$$

we have that $w^m = \xi(w)$, a sum of D-monomials in w of degree less than m. It follows easily that

$$v = s^m - \xi(s) = \begin{pmatrix} 0 & 0 & 0 \\ * & c & 0 \\ * & * & 0 \end{pmatrix}.$$

Now choose

$$t = \begin{pmatrix} * & 0 & 0 \\ * & c & 0 \\ * & * & * \end{pmatrix} \in D,$$

so that

$$v^2 - tv = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{bmatrix}.$$

This gives $(v^2 - tv)^2 = 0$. Substituting back, we get that a satisfies a monic polynomial of degree $4(m+2)m$. ■

We now record a key lemma from [Q2].

LEMMA 2 [Q2, Proposition 1.2]. *Let $T \supseteq R \supseteq S$ be rings with R Schelter integral (resp., fully integral of degree m) over the subring S. Suppose further that $e \in T$ is an idempotent such that $eRe \subseteq R$. Then eRe is Schelter integral (resp., fully integral of degree m) over eSe.* ■

We now prove that subnormalizing extensions are fully integral. The proof here is essentially the same as the Lorenz-Passman proof [LP, Corollary 2], once we have Theorem 1. If $S = Rx_1 + Rx_2 + \cdots + Rx_n$ is a finite subnormalizing extension, an ideal I of R is called subnormal if $Ix_1 + Ix_2 + \cdots + Ix_i = x_1I + x_2I + \cdots + x_iI$, for $i = 1,2,\ldots,n$.

THEOREM 3. *Suppose $S = Rx_1 + Rx_2 + \cdots + Rx_n$ is a finite subnormalizing extension of R, and let I be a subnormal ideal of R. Then $SI = IS$ is fully integral over I of degree $m(n)$.*

PROOF. We will just sketch the proof and refer the reader to [LP] for complete details. Let $F = R^{(n)}$ be a direct sum of n copies of R and map F to S by the map π, sending $(r_1, r_2, \ldots, r_n) \in F$ to $\sum_i r_i x_i \in S$. Note that this map is clearly onto and let K denote the kernel. Let $A = \{\phi \in M_n(R) \mid K\phi \subseteq K\}$. A is a subring of $M_n(R) = \mathrm{End}_R F$. There is an obvious ring homomorphism from A to $\mathrm{End}_R S$, the endomorphisms of S as a left R-module.

Let D be the set of lower triangular matrices $(\alpha_{i,j}) \in M_n(I)$ such that there exists $t \in I$ with $\sum_{j=1}^{i} \alpha_{i,j} x_j = x_i t$, for all i. It can be easily checked that D is a subring of $M_n(I)$ that is contained in A. Since I is a subnormal ideal, the image of D in $\mathrm{End}_R S$ is precisely the set of multiplications by elements of I, and $e_{i,i} D e_{i,i} = I e_{i,i} \subseteq M_n(I)$, for each i.

Now let $s \in SI = IS$, which is a two-sided ideal of S. Thus we can find a matrix $\beta = (\beta_{i,j}) \in M_n(I)$ such that $\sum_{j=1}^{n} \beta_{i,j} x_j = x_i s$, for all i. Then $\beta \in A$ and the image of β in $\mathrm{End}_R S$ is right multiplication by s.

We now have that the image of $M_n(I) \cap A$ contains the set of right multiplications by elements of IS. By Theorem 1, $M_n(I)$ is fully integral over D and the result follows by considering images in $\mathrm{End}_R S$. ∎

Restricted Lie algebras and crossed products

Since a crossed product of the restricted enveloping algebra of a finite dimensional restricted Lie algebra is a finite subnormalizing extension of the coefficient ring, as an immediate consequence of the last result we get the following corollary. The reader is referred to [C1] for details of the construction of these crossed products.

COROLLARY 4. *Suppose L is a finite dimensional restricted Lie algebra with restricted enveloping algebra $u(L)$ and let $R * u(L)$ be a crossed product of $u(L)$ over R. Then $R * u(L)$ is fully integral over R.* ∎

Another consequence of Theorem 3 is the following Going Up theorem for restricted enveloping algebra crossed products. This is analogous to the Going Up theorem for crossed products over finite groups [LP, Theorem 5].

COROLLARY 5. *Let $S = R * u(L)$ be a crossed product of the restricted enveloping algebra $u(L)$ of a finite dimensional restricted Lie algebra L. Suppose that Q is an L-prime ideal of R and that P is a prime ideal of S with $P \cap R \subset Q$. Then there exists a prime ideal P' of S with $P \subset P'$ and $P' \cap R = Q$.*

PROOF. Since Q is L-stable, Q is a subnormal ideal of R. Passing to $\bar{S} = S/P$, $\bar{S}$ is again a subnormal extension of $\bar{R}$, the image of R. Note that $\bar{Q}$ is a subnormal ideal of $\bar{R}$. By Theorem 3, $\bar{Q}\bar{S}$ is fully integral over $\bar{Q}$. Thus we have that $\bar{Q}\bar{S} \cap \bar{R}$ is nilpotent modulo $\bar{Q}$. Since $P \cap R \subset Q$, taking preimages in $S = R * u(L)$, it follows that $QS + P$ is an ideal of S, with $(QS + P) \cap R$ nilpotent modulo Q. But since $(QS + P) \cap R$ is an L-invariant ideal of R and Q is L-prime, we have that $(QS + P) \cap R = Q$. Thus we have $J = QS + P$, an ideal of S with $J \cap R = Q$ and $P \subset J$. Now J can be extended by Zorn's lemma, to an ideal P' of S which is maximal with respect to $P' \cap R = Q$. It is standard and easily checked that P' is a prime ideal. ∎

Prime ideals in these crossed products were studied in [C1]. When L is abelian and finite dimensional, incomparability and Going Down were proved for the extension $R \subset S$.

Algebraic automorphisms

We next turn our attention to automorphisms and fixed rings.

PROPOSITION 6. *Let R be an algebra over a commutative ring k. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be k-automorphisms of R and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in k$ such that $\sum_i \alpha_i = 1$. If I is an ideal of R such that $\sigma_i(I) = I$ for each i, then I is fully integral over J, the image of I under $\sum_i \alpha_i \sigma_i$.*

PROOF. Let $D = \{\mathrm{diag}(\sigma_1(r), \sigma_2(r), \ldots, \sigma_n(r)) \mid r \in I\} \subseteq M_n(I)$ and let e be the idempotent matrix

$$e = \begin{bmatrix} \alpha_1 & \alpha_1 \cdots \alpha_1 \\ \alpha_2 & \alpha_2 \cdots \alpha_2 \\ \alpha_n & \alpha_n \cdots \alpha_n \end{bmatrix}.$$

By Theorem 3, $M_n(I)$ is fully integral over D, and we can then apply Lemma 2 to conclude that $eM_n(I)e = Ie$ is fully integral over $eDe = Je$. But now Ie is isomorphic to I by an isomorphism taking Je to J. ■

THEOREM 7. *Let R be an algebra over the field k and suppose that σ is an algebraic k-automorphism of R with minimal polynomial $p(x) \in k[x]$. Let $p(x) = (x-1)^t g(x)$, where $g(1) \neq 0$. Then R is fully integral over $\{r \in R \mid (\sigma - 1)^t(r) = 0\}$. In particular, if $t = 1$, then R is integral over the fixed ring $R^{\langle\sigma\rangle}$.*

PROOF. Let $g(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_t x^t$. Since $g(1) \neq 0$, we can assume that $\sum_i \alpha_i = 1$. Now apply the previous result with $\sigma_i = \sigma^i$. Clearly, the image of $\sum_i \alpha_i \sigma^i$ is contained in $\{r \in R \mid (\sigma - 1)^t(r) = 0\}$. The last statement is clear. ■

The following result combines the result of Bergman (see [P1]) for graded rings together with integrality for fixed rings. We comment that an example of Bergman shows that in the noncommutative setting, integrality is not transitive. Thus it is not possible to simply quote those results.

THEOREM 8. *Let R be graded by the finite group G and let H be a finite group of automorphisms of R that permute the homogeneous components with $|H|$, the order of H, invertible. Then R is fully integral over R_1^H of degree $m(|H| \times |G|)$.*

PROOF. As in [Q1], define $R\#G^* = \sum_{g,h \in G} R_{gh^{-1}} e_{g,h} \subseteq M_G(R)$. If we let $T = M_H(R\#G^*)$, then T can be regarded as a subring of $M_{G \times H}(R)$. If $x, y \in H$, let $E_{x,y} \in M_H(R\#G^*) \subseteq M_{G \times H}(R)$ be the element with $1 = 1_{R\#G^*}$ in the (x,y)-

position and zeros elsewhere. Let $A \subseteq T$ be given by $A = \{\sum_{h\in H} r^h E_{h,h} \mid r \in R_1\}$. Viewing $A \subseteq T \subseteq M_{G\times H}(R)$, we see that A is a subring of diagonal matrices, so that it follows from Theorem 1 that T is fully integral over A of degree $m(|H| \times |G|)$. Now let $f \in T \subseteq M_H(R\#G^*)$ be the element with

$$\frac{1}{|H|} = \frac{1}{|H|} 1_{R\#G^*}$$

in every position. Since $f \in T$ is an idempotent, we can apply Lemma 2 to conclude that fTf is fully integral over fAf. It is easily checked that $fTf = (R\#G^*)f$ is isomorphic to $R\#G^*$ by an isomorphism which takes fAf to R_1^H. Since $R \subseteq R\#G^*$, the result follows. ∎

Of course, the proof of the last result showed that $R\#G^*$, and not just R, is integral over R_1^H. That $R\#G^*$ is integral over R_1 is implicit in [P1]. A small modification could be made so that the theorem included Theorem 7.

Example 9. (i) Suppose the finite group G acts on R and $|G|$ is invertible in R. Then the skew group ring RG is integral over R^G.

This follows immediately from the theorem, since G acts as graded automorphisms of RG.

(ii) Let R be G-graded with $|G|^{-1} \in R$. Then $M_G(R)$ is fully integral over R_1.

To see this, first observe that $M_G(R)$ can be graded by the group $G \times G$, where $G \times 1$ grades R and the grade $(1, gh^{-1}) \in G \times G$ is assigned to the matrix unit $e_{g,h}$. The identity component of $M_G(R)$ is the set of diagonal matrices with entries from R_1. Now the group of permutation matrices $\Gamma = \{\bar{g} = \sum_{x\in G} e_{x,xg} \mid g \in G\}$ acts by conjugation, as graded automorphisms of $M_G(R)$, and the identity component is R_1. ∎

Hopf algebra actions

Let H be a finite dimensional semisimple Hopf algebra and R an H-module algebra. In [Q2] we raised the question as to whether R is integral over the invariant subring R^H. The answer is known to be "yes" when H is either a group algebra [Q2] or its dual; see [P1]. In the case where H is the dual of a group algebra, R is simply a graded ring. Here the invariant subring is R_1, the identity component of R. There is also a positive answer [Q2] if the action of H is inner in the sense of [BCM]. An earlier proof in the case of an inner action on an H-prime ring R is due to M. Cohen [Co]. Here we give another case which follows from Theorem 8.

THEOREM 10. *Suppose H is a finite dimensional cocommutative semisimple Hopf algebra over an algebraically closed field k, of positive characteristic p. Let R be an H-module algebra. Then R is fully integral over R^H.*

PROOF. By Kostant's theorem [S1, 8.1.5], $H = A\#k[G]$ is a smash product, where A is an irreducible Hopf subalgebra and $k[G]$ is the group algebra of the set of group-like elements of H. A is a semisimple cocommutative irreducible Hopf algebra. From [S2, Theorem 4.1] we know that A is the dual of the group algebra of a p-group P. Since A acts on R, we get that R is graded by the group P. The elements of G act on A as automorphisms and hence permute the primitive idempotents corresponding to the elements of P, which span A. See [CM] for details. Suppose $x \in P$, $g \in G$ and p_x is the primitive idempotent of A corresponding to x. Then in H, $gp_xg^{-1} = p_y$, for some $y \in P$. Now $gR_x = gp_xR = p_ygR = R_y$. In other words, G acts on R as graded automorphisms. Furthermore, $k[G]$ is also semisimple so that $|G|$ is a unit in k by Maschke's theorem. Thus the invariant subring R^H is the set of fixed elements in R_1 so that the result now follows from Theorem 8. ∎

An elementary proof of the result of Sweedler [S2, Theorem 4.1] in the finite dimensional case we need can be found in [C2].

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